Downstream Boundary Conditions for Vertical Jets

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One of the more important fluid mechanics problems with a free surface is the analysis of the laminar flow of an incompressible Newtonian liquid jet issuing from a circular nozzle into an inviscid gas phase. Considerable effort has been expended in recent years in obtaining numerical solutions to the complex set of nonlinear equations describing this flow field. Duda and Vrentas (1967) studied the high Reynolds numbers limit where the boundary layer assumptions are valid and the governing differential equations are parabolic. Finite-difference solutions were obtained for both horizontal and vertical jets with and without surface tension using a nonorthogonal Protean coordinate system with the stream function as one of the independent variables. Numerical solutions of the elliptic partial differential equations describing jet fluid dynamics were obtained by Horsfall (1973), Nickell et al. (1974), Reddy and Tanner (1978), Chang et al. (1979), and Omodei (1980). Results were reported for a wide Reynolds number range for horizontal jets, and surface tension effects were considered. Dutta and Ryan (1982) presented results for creeping horizontal and vertical Newtonian jets with and without surface tension. A finite-difference technique was used, and an orthogonal coordinate system was introduced to circumvent difficulties introduced by the presence of a free boundary. Finally, Fisher et al. (1980) used a finite element method to calculate the velocity field for a vertical free Newtonian jet.

One important aspect of the analysis of liquid jets is the problem of formulating appropriate boundary conditions far downstream from the jet nozzle. This problem is avoided in the high Reynolds number limit (Duda and Vrentas, 1967) since the governing equations are parabolic and only initial conditions are needed in the principal flow direction. However, at all other Reynolds numbers, the equations are elliptic and downstream boundary conditions are required for a properly posed problem. For horizontal jets, it is generally accepted that a radially uniform axial velocity distribution and a final jet diameter are approached asymptotically as the distance from the jet nozzle is increased. In this case, all velocity gradients vanish in the infinite limit. For vertical jets, however, at least two types of downstream boundary conditions have been proposed. The purpose of this note is to discuss the formulation of downstream boundary conditions for vertical Newtonian jets and to present some selected results for liquid jets under the influence of gravity.

DOWNSTREAM BOUNDARY CONDITIONS

In practice, the mechanics of a vertical Newtonian jet will depend on the precise geometry of the flow, as characterized, for example, by the distance from the nozzle to a solid object upon which the jet impinges. However, if all obstacles are far from the nozzle, it seems reasonable to assume that the effect of distant boundaries is insignificant and, hence, to suppose that the jet extends to infinity with the speed of flow approaching that of a freely falling body. Even for jets with very low Reynolds numbers (based on conditions at the jet nozzle), it is presumed that, for a long enough jet, the inertia term will eventually grow so that it overwhelms viscous forces and balances the gravitational term, thus leading to a free-fall velocity field.

Various aspects of this point of view have been considered by Scriven and Pigford (1959), Brown (1961), Duda and Vrentas (1967), Clarke (1968, 1969), Bentwich (1970), and Kistler and Scriven (1981). Brown collected velocity data for a falling liquid curtain and concluded that a modified free-fall equation represented the data quite well at sufficiently great jet lengths, even though nozzle Reynolds numbers as low as about 0.05 were utilized. Comparison of the data with the theory was carried out at jet lengths ranging from approximately 30 to 250 slot widths. Clarke (1968) proposed that the asymptotic solution valid far downstream (the outer solution) be matched with an upstream solution (the inner solution), with the outer velocity field approaching the free-fall velocity at large enough distances. Kistler and Scriven implemented such a matching procedure for a falling liquid curtain by matching finite element solutions of the complete set of equations with a one-dimensional asymptotic approximation that ultimately achieves a free gravity flow field. It is evident from the above discussion that the utilization of a free-fall velocity field as a downstream boundary condition for a vertical jet is a reasonable hypothesis that has some experimental backing. It is, of course, necessary to point out that the velocity field computed from this approach may not actually be observed experimentally because of instabilities promoted by surface tension effects.

A different set of downstream boundary conditions has been formulated by Dutta and Ryan (1982) for a vertical jet. They propose that downstream boundary conditions for a vertical jet be imposed at a finite jet length (equal to one nozzle diameter). It appears that Dutta and Ryan are proposing that the downstream boundary conditions for a vertical jet are effectively achieved at a finite jet length and are maintained for all greater jet lengths, since, otherwise, it is not in general possible to introduce sufficiently detailed conditions on the velocity field at a finite length for a jet that is presumably of infinite length. Possible difficulties are illustrated by the results of Fisher et al. (1980), who formulated downstream boundary conditions for a vertical jet at a finite jet length and showed that the computed velocity field was strongly dependent on the jet length at which these conditions are imposed.

Dutta and Ryan stated, without discussion, that the axial velocity profile was uniform at the chosen jet length, that the vorticity was uniformly zero, and that the radial velocity at the jet surface was zero. These conditions are then presumably valid for all greater jet lengths, and a final jet diameter can be computed. If these conditions are imposed, it follows that the equations of motion for large distances from the nozzle reduce to

$$\frac{\partial p}{\partial r} = 0 \tag{1}$$

$$\frac{\partial p}{\partial z} = \rho g \tag{2}$$

These equations imply that the radially uniform pressure in the jet increases linearly with z at sufficiently large jet lengths. On the other hand, in the absence of surface tension, one of the surface boundary conditions requires that

$$p = 0 \tag{3}$$

everywhere along the jet surface at sufficiently large downstream positions if the gas pressure is set equal to zero. Hence, Eqs. 1 and 3 require that the pressure is zero everywhere at sufficiently large jet lengths, a result inconsistent with the linear increase in pressure demanded by Eqs. 1 and 2. The approach proposed by Dutta and Ryan thus appears to lead to inconsistent requirements on the pressure field. The gravitational force must be balanced in the jet, but, in the absence of velocity gradients, inertia and viscous forces are both zero. Hence, the pressure field is the only candidate, but it cannot balance gravity in a consistent manner. From this discussion, we conclude that downstream boundary conditions formulated by requiring that all velocity gradients effectively vanish at a finite jet length do not lead to an internally consistent flow field for a vertical jet.

RESULTS

To illustrate the roles of inertia, viscous forces, and gravity in an axisymmetric, vertical Newtonian jet issuing into an inviscid gas phase, we present some selected computations of the dependence of jet radius on axial position. These calculations were carried out using a Protean coordinate system (Duda and Vrentas, 1967) and matching of a two-dimensional finite-difference solution with an asymptotic one-dimensional approximation. It can be easily shown

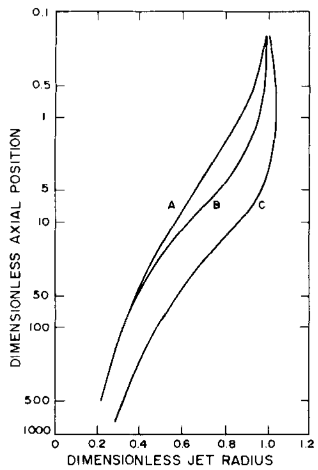

Figure 1. Dependence of dimensionless jet radius on dimensionless axial distance for axisymmetric vertical jet. Curve A: *Re* = 50, *Fr* = 1; curve B: *Re* = 1, *Fr* = 1; curve C: *Re* = 1, *Fr* = 5.

that the radius of a vertical jet of an incompressible Newtonian fluid is represented by an equation of the following functional form:

$$\frac{R}{R_o} = F\left(\frac{z}{R_o}, Re, Fr\right) \tag{4}$$

when surface tension effects are neglected. Hence, in Figure 1, results are presented for the dependence of the dimensionless radius on dimensionless axial distance for particular values of Re and Fr.

For case A ($Re = 50, Fr = 1$), inertia and gravity dominate the jet, and this results in a rapid contraction since viscous effects play a relatively minor role. For case B ($Re = 1, Fr = 1$), viscous effects are considerably more important as the creeping flow region is approached, and appreciable contraction of the jet occurs only at significantly larger jet lengths than in case A. However, the inertia and gravitational forces eventually dominate, and the jet shape is independent of Re at a dimensionless axial position of approximately 100. For case C ($Re = 1, Fr = 5$), the gravitational force is weaker than in cases A and B, and viscous effects close to the nozzle actually lead to a slight expansion of the jet before contraction toward the point sink at infinity begins as inertia and gravity eventually overwhelm the viscous force. Both Re and Fr have a strong influence on the jet shape near the nozzle, but only Fr has an effect on the jet radius sufficiently far away from the nozzle. Although only selected results are presented here, we believe that these represent the first correct solutions of the complete equations of motion for circular, Newtonian vertical jets.

NOTATION

Fr	= Froude number = $U_a^2/2R_o g$
g	= acceleration of gravity
p	= pressure
r	= radial distance variable
R	= radius of jet
R_o	= nozzle radius
Re	= Reynolds number = $2R_o\rho U_a/\mu$
U_a	= average axial velocity at nozzle
z	= axial distance variable

Greek Letters

μ	= viscosity of liquid
ρ	= density of liquid

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